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On S-matrix factorization of the Landau-Lifshitz model

A. Melikyan, A. Pinzul, V.O. Rivelles and G. Weber

Instituto de Física, Universidade de São Paulo,
C. Postal 05315-970, São Paulo, SP, Brazil
E-mail: `melikyan@fma.if.usp.br`, `apinzul@fma.if.usp.br`, `rivelles@fma.if.usp.br`, `weber@fma.if.usp.br`

Abstract: We consider the three-particle scattering S-matrix for the Landau-Lifshitz model by directly computing the set of the Feynman diagrams up to the second order. We show, following the analogous computations for the non-linear Schrdinger model [1, 2], that the three-particle S-matrix is factorizable in the first non-trivial order.

Keywords: Exact S-Matrix, Bethe Ansatz, Integrable Field Theories, Sigma Models.

Contents

1. Introduction

The Landau-Lifshitz (LL) model describes the dynamics of the classical Heisenberg spin chain, and has appeared in the past few years in both gauge and string theories as one of the first indications of the underlying integrable structure [3 − 15] of the AdS/CFT correspondence (for a review see [16 − 19]). On the string theory side the LL model emerges in the $R \times S^3$ subsector in the limit of large angular momentum [3, 20 − 25]. Alternatively, the LL model can be derived from the Faddeev-Reshetikhin (FR) model in the low-energy limit [26, 27]. The latter is aimed to resolve the difficulties of the quantization procedure [28 − 36] inherent to all sigma models. More recently, the LL model also emerged in the SU(2) × SU(2) subsector of the strings on $AdS_4 \times \mathbb{C}P^3$ [37] in the context of the newly proposed duality between $\mathcal{N} = 6$ superconformal Chern-Simons and the $AdS_4 \times \mathbb{C}P^3$ string theories in the t'Hooft limit [38]. The integrable properties of the LL model have been extensively discussed from various points of view. The classical integrability was established for the isotropic case in [39], and for the general anisotropic case in [40, 29], and the quasi-classical analysis was performed in [41]. The classical equivalence between the LL and the non-linear Schrdinger (NLS) models was shown in [42] by constructing a gauge transformation between the flat currents of the corresponding models.

At the quantum level, on the other hand, the integrable properties of the LL and NLS models are quite different. In [43] Sklyanin has considered in detail the quantum inverse scattering method for the LL model (the $su(1,1)$ case), and pointed out several subtleties of the quantization procedure, absent in the classically equivalent NLS model. One of the surprising difficulties is that the standard methods to obtain the Yang-Baxter relation from its classical counterpart fail for the anisotropic LL model, and the algebra of observables, as well as the transfer matrix have to be modified by hand in order to construct the quantum R-matrix. More important implication of the Sklyanin's analysis is the problem of constructing the local integrals of motion in the quantum theory, underlying the quantum integrability of the model. In fact, the problems already arise in constructing the local quantum Hamiltonian for the two-particle sector, and to the best of our knowledge, the higher order charges have not been constructed so far. The difficulties arise as a result of the ill-defined operator product at the same point[1] and can, in principle, be resolved by the standard procedure of putting the model on the lattice [45, 46]. Although this can be done for both FR and LL models [26, 47, 48], and the corresponding spectra can be found with the standard Bethe Ansatz technique, the full $AdS_5 \times S^5$ string model is too complicated, and in general putting a continuous theory on the lattice using the standard methods leads to a non-local quantum Hamiltonian [45, 49]. Thus, one has to deal with the quantization of the continuous models directly, and the LL model, despite its simplicity, is a particularly interesting representative example of the difficulties associated with the continuous quantum inverse scattering method.

The LL and FR models, as well as a number of other models, arising in various limits of the full $AdS_5 \times S^5$ string, have recently been considered as two-dimensional field theories, and the corresponding S-matrices have been obtained by means of perturbative calculations [27, 50 – 53]. The idea of this method and its relation to the inverse scattering method is nicely reviewed for the NLS model in [54]. This method makes use of the integrability, which is believed to hold at the quantum level, and, in particular, the S-matrix factorization property [55 – 62], which renders the N-particle scattering S-matrix to be a product of the two-particle S-matrices. The direct verification of the factorization property, by calculating the set of Feynman diagrams, has only been possible to carry out for the NLS model in [1, 2]. More recently, the three-particle S-matrix factorization property was demonstrated for the $AdS_5 \times S^5$ string in the near flat space limit at one-loop order [53].

In this article we consider the three-particle scattering process for the LL model and, by computing the necessary Feynman diagrams up to the second order, show that it is factorizable in the first non-trivial order. There are several conceptual features that make the S-matrix calculations for the LL model a non-trivial task. First, as we noted earlier, the LL model is the low energy limit for the FR model, and as was shown in [36], the Hamiltonian for the FR model is not diagonalizable in the class of the standard representation for the two-particle sector:

$$|p_1 p_2\rangle = \iint\limits_{x_1 \neq x_2} dx_1 dx_2 \left[e^{p_1 x_1 + p_2 x_2} + (S)\, e^{p_1 x_2 + p_2 x_1}\right] \phi^+(x_1)\phi^+(x_2)|0\rangle \tag{1.1}$$

This representation is at the heart of the Bethe Ansatz and has the clear interpretation for the first and second terms as the incoming and outgoing waves respectively, and where the S-matrix $S(p_1, p_2) = e^{i\Delta(p_1,p_2)}$ is a phase-shift due to the scattering (we assume that $p_1 > p_2$). Proceeding in a similar manner to [36] it is not difficult to show that

[1] For more details see the upcoming paper [44]

the Hamiltonian for the LL model cannot be diagonalized for the class of functions (1.1), although the exact two-particle scattering S-matrix for the FR model can be obtained using the perturbative calculations [27]. Alternatively, this can also be seen directly via the quantum inverse scattering method [43], where the two-particle wave function is shown to acquire an additional term to (1.1). One of the reasons why this happens is the extremely singular nature of the LL interaction [43]. In fact, it is not difficult to see that the interaction in the quantum mechanical picture corresponds to the second derivative of the delta-function, and dealing with it is not an easy task (see for example [63]). In order to write the additional term to (1.1), one has to introduce a new creation operator, which for the two-particle state would correspond to a bound state. It is not difficult to see that the number of such terms increases with the number of particles in the scattering process. For example, in the three-particle scattering process there will be two additional contributions, corresponding to three and two-particle clusters.

More importantly, the form (1.1) suggests that the particles created by the fields $\phi^+(x)|0\rangle$ are those corresponding to the Bethe particles, and all the consequences of the integrability, such as the S-matrix factorization, will hold as the consequence of the general argument of [55, 56, 59]. This is indeed the case for a number of models, e.g. the NLS model, where (1.1) is indeed the two-particle state, and the particles $\phi^+(x)|0\rangle$ are the Bethe particles, for which one can obtain the S-matrix either using the perturbative diagrammatic calculations, or via direct diagonalization of the Hamiltonian. However, this *a priori* is not the case in general, and the particles created by the fields $\phi^+(x)$ may not correspond to the Bethe particles. This is the case for the LL model, namely, the Bethe particles do not coincide with $\phi^+(x)|0\rangle$. Let us emphasize that considering the scattering process and stating the S-matrix factorization makes sense only in terms of the Bethe particles. The only reliable method for diagonalizing all the local conserved quantities simultaneously, including the Hamiltonian, and, as a consequence, constructing the eigenstates corresponding to the Bethe particles, is the quantum inverse scattering method. But as we mentioned above, this construction fails in the LL model, and the local conserved quantities cannot be derived from the trace identities due to the singular expressions. Thus, checking the quantum integrability, e.g., the factorization of the S-matrix, as well as establishing a connection with the construction in the quantum inverse scattering method by using the standard field theoretic methods is an important task that should prove useful when considering more complicated systems than the LL model. In this paper and [44] we make the first steps towards this program, and as the first result we show here the factorization in the lowest order.

There are other subtleties, making factorization for the LL model a non-trivial feature. The unconstrained two dimensional field theory on the world-sheet corresponding to the LL model contains infinite number of vertices. This is in contrast with the simpler NLS model, where there is simply one vertex to deal with, and, thus, making the higher order calculations much simpler. In the LL model one has to correspondingly consider new types of vertices as the order of calculations increases. Let us recall, that to calculate the two-particle scattering S-matrix one only needs to keep the terms up to the quartic order in the Lagrangian. The role of higher order terms is to preserve the integrability at the quantum level. Namely, for the three-particle scattering S-matrix one has to consider the vertices of higher (up to ϕ^6) order. The quantum integrability manifests here in the S-matrix factorization for the three-particle scattering. Although the combinatorial analysis due to the infinite number of vertices makes practically impossible to carry out the calculations to all orders, we are able to show in the first non-trivial order that the higher order vertices are exactly of the form needed for the S-matrix to be factorizable. The mechanism behind factorization is not straightforward, and we show that the higher order terms cancel unwanted lower order terms to guarantee the factorization. Moreover, we show explicitly that there is no process corresponding to particle annihilation or creation, and the set of momenta before and after the scattering is the same, in agreement with our understanding of integrability.

Our paper is organized as follows. In section 2, we give a brief account of the perturbative S-matrix calculations for the Landau-Lifshitz model in the two-particle sector. In section 3, we present our analysis of the three-particle scattering for the LL model and show the factorization in the first non-trivial order. In section 4, we give a brief summary of our results. In the appendix we collect the set of the Feynman graphs used in our calculations in order to avoid cluttering the main text.

2. Two-particle scattering S-matrix

In this section we set up the notation and present the necessary formulas for the subsequent section on the three-particles scattering. First we briefly review the necessary facts about the LL model, and then explain the two-particle perturbative S-matrix calculations, referring the reader to [27] for complete details of the calculations. Let also us note, that the two-particle scattering diagrammatic calculations for the LL model are quite similar to those of the non-linear Schrdinger model [54]. Indeed, the only essential difference is the presence of the derivatives in the interaction terms. In the context of the perturbative calculations this results in a slightly more complex combinatorial analysis in the LL model. As we will see below, the essential differences arise when considering the three-particle scattering amplitude.

2.1 Landau-Lifshitz model: preliminary facts

The Landau-Lifshitz model is a non-relativistic sigma model which describes the continuous Heisenberg ferromagnet, and arises in the $R \times S^3$ subsector of the $AdS_5 \times S^5$ strings in the limit of the large angular momentum. The equations of motion for the isotropic LL model, on which we will focus in this paper, in terms of the spin variables $\mathbf{S}(x) = (S^a(x); a = 1, 2, 3)$, have the form:

$$\partial_t \mathbf{S} = \mathbf{S} \times \partial_x^2 \mathbf{S} \tag{2.1}$$

where the fields $\mathbf{S}(x)$ take values on S^2:

$$\mathbf{S}^2 = 1 \tag{2.2}$$

The equation (2.1) can be obtained from the Hamiltonian:

$$H_{\mathrm{LL}} = \frac{1}{4} \int dx (\partial_x \mathbf{S})^2 \tag{2.3}$$

with the Poisson structure:

$$\{S^a(x), S^b(y)\} = -\varepsilon^{abc} S^c \delta(x-y) \tag{2.4}$$

In the most general formulation, the equations of motion for the anisotropic LL model are modified by an anisotropy matrix $\mathbf{J} = \mathrm{diag}(J_1, J_2, J_3)$, and take the form:

$$\partial_t \mathbf{S} = \mathbf{S} \times \partial_x^2 \mathbf{S} + \mathbf{S} \times \mathbf{J}\mathbf{S} \tag{2.5}$$

As explained in [43] the isotropic and anisotropic cases should be considered separately when quantizing the theory. In the former case, the standard inverse scattering procedure goes through without any changes, and the Yang-Baxter equation is satisfied with the appropriate choice of the R-matrix, while in the later case the operators, and accordingly the operator algebra (2.4) have to be modified by hand for the Yang-Baxter relation to have a solution. In this paper we will consider only the isotropic case $\mathbf{J} = \mathbf{0}$, and the more general case is currently under investigation.

The corresponding action can be written in a manifestly covariant form as follows:

$$S = \int d^2x \left[C_t(\mathbf{S}) - \frac{1}{4} (\partial_x \mathbf{S})^2 \right] \tag{2.6}$$

where $C_t(\mathbf{S})$ is the Wess-Zumino term:

$$C_t(\mathbf{S}) = -\frac{1}{2} \int_0^1 d\xi \; \epsilon_{ijk} S_i \partial_\xi S_j \partial_t S_k \tag{2.7}$$

The boundary conditions for the $\mathbf{S}(t, x; \xi)$ field have the form:

$$\begin{cases} \mathbf{S}(t, x; \xi = 1) = & \mathbf{S}_0 \\ \mathbf{S}(t, x; \xi = 0) = & \mathbf{S}(t, x) \end{cases} \tag{2.8}$$

where $\mathbf{S}_0$ is a constant vector.

Following [27], it is convenient to resolve the constraint (2.2) and write the action (2.6) as an unconstrained $(1+1)$-dimensional quantum field theory. This can be achieved by a change of variables, which get rid of the non-linearities in the kinetic term, as follows

$$\varphi = \frac{S_1 + i S_2}{\sqrt{2 + 2S_3}} \quad , \quad S_3 = 1 - 2|\varphi|^2 \tag{2.9}$$

and which turns the Wess-Zumino term (2.8) into a but non-covariant form. The resulting action following from (2.6) and (2.9) can be written as follows:

$$\begin{aligned} S = \int d^2x \, \Bigg\{ & \frac{i}{2} (\varphi^* \partial_t \varphi - \partial_t \varphi^* \varphi) - |\partial_x \varphi|^2 \\ & - \frac{1}{4} \frac{2 - |\varphi|^2}{1 - |\varphi|^2} \left[(\varphi^* \partial_x \varphi)^2 + (\partial_x \varphi^* \varphi)^2 \right] - \frac{1}{2} \frac{|\varphi|^4 |\partial_x \varphi|^2}{1 - |\varphi|^2} \Bigg\} \end{aligned} \tag{2.10}$$

The non-relativistic character of the LL model leads to significant simplifications in the quantization procedure. In particular, it makes diagrammatic calculations quite easy

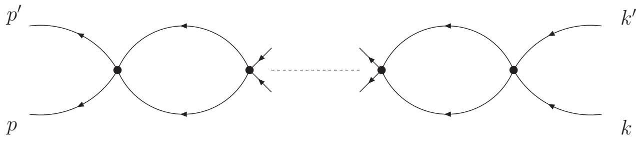

Figure 1: Typical bubble diagram for the two-particle S-matrix.

to deal with. This is mostly due to the property of the propagator in the LL model to be a retarded one. Let us also note, that in relativistic theories, such as the massive Thirring model, to have similar simplifications in diagrammatic calculations, one can choose the false vacuum by hand, and, therefore, making the Feynman propagators to be retarded.

Using the field decomposition

$$\varphi(x) = \int \frac{dp}{2\pi} e^{-ip\cdot x} a_p \quad , \quad \varphi^*(x) = \int \frac{dp}{2\pi} e^{ip\cdot x} a_p^\dagger \tag{2.11}$$

where we use the notation $p \cdot x = p^2 x^0 - px$, and defining the vacuum of the theory by $\varphi(x)|0\rangle = 0$, the annihilation and creation operators a_p and $a_p^\dagger$ satisfy the standard commutation relation:

$$[a_{p'}, a_p^\dagger] = 2\pi\delta(p - p'). \tag{2.12}$$

One then finds the propagator to be of the form [27]:

$$\begin{aligned} D(t,x) &= \int \frac{d^2q}{(2\pi)^2} \frac{ie^{-iq\cdot x}}{q^0 - q^2 + i\epsilon} \\ &= \theta(t)\sqrt{\frac{\pi}{it}} e^{\frac{ix^2}{4t}} \end{aligned} \tag{2.13}$$

The fact that the propagator is purely retarded leads to the following result: the two particle scattering S-matrix is given by the sum of bubble diagrams, as in figure 1.

The diagram above is exactly the one that appears in the NLS model. For the LL model one has to properly take into account the presence of the derivatives in the interaction terms, which results in four different types of vertices, depending on the placement of the derivatives on the internal or external lines. Nevertheless, the calculations are still easy to carry out.

2.2 Diagrammatic calculations

Since the only contributions to the two-particle scattering amplitude are the diagrams of the type figure 1, to calculate the two-particle S-matrix it is enough to truncate the complicated action (2.10) up to the fourth order in the fields:

$$S = \int d^2x \, \frac{i}{2}(\varphi^*\partial_t\varphi - \partial_t\varphi^*\varphi) - |\partial_x\varphi|^2 - \frac{g}{2}[(\varphi^*\partial_x\varphi)^2 + (\partial_x\varphi^*\varphi)^2] + O(\varphi^6) \tag{2.14}$$

We introduced in the expansion (2.14) the formal parameter g to keep track of the perturbative order, and it should be set to one at the end of the calculations. Let us note here, that this is very important for the three-particle scattering amplitude factorization, where several such formal parameters will be introduced to take care of order counting for different types of vertices that arise in the LL model when expanding the action (2.10) up to the sixth order in the fields. A natural question arises whether setting the parameters to one at the end is consistent with the renormalization properties of the LL model. Although we ignore here all the divergences associated with the loop calculations, this is an interesting question and is currently under investigation.

We collect here the necessary expressions, referring the reader to [27] for complete details. Taking without any loss of generality the ordering $p > p'$ for the scattering to take place, the two-particle S-matrix is determined from the relation

$$\langle kk'|\hat{S}|pp'\rangle = S(p,p')\delta_+^{(2)}(p,p';k,k') \tag{2.15}$$

where:

$$\delta_+^{(2)}(p,p';k,k') = (2\pi)^2\left[\delta(p-k)\delta(p'-k') + \delta(p-k')\delta(p'-k)\right] \tag{2.16}$$

The scattering amplitude given by:

$$\begin{aligned}\langle kk'|\hat{S}|pp'\rangle &= \langle kk'|Te^{-i\int H_{\rm int}dt}|pp'\rangle \\ &= \langle kk'|pp'\rangle - i\langle kk'|T\int H_{\rm int}dt|pp'\rangle - \frac{1}{2}\langle kk'|T\left(\int H_{\rm int}dt\right)^2|pp'\rangle + \cdots \end{aligned} \tag{2.17}$$

is easily computed with $H_{\rm int} = \frac{g}{2}\int dx\ \left[(\varphi^*\partial_x\varphi)^2 + (\partial_x\varphi^*\varphi)^2\right]$ in each order. The non-scattering part and the tree level parts are easily computed and have the forms:

$$\langle kk'|pp'\rangle = \delta_+^{(2)}(p,p';k,k') \tag{2.18}$$

$$-i\langle kk'|T\int H_{\rm int}dt|pp'\rangle = 2ig\frac{pp'}{p-p'}\delta_+^{(2)}(p,p';k,k') \tag{2.19}$$

To avoid the combinatorial analysis of [27], associated with placement of derivatives on internal and external lines, we will compute the full bubble diagram in the following manner. Using the complete interaction vertex, which in the momentum representation has the form:

$$V(k,k';p,p') = 2ig\left[pp'+kk'\right]\delta(p^0+p'^0-k^0-k'^0)\delta(p+p'-k-k'), \tag{2.20}$$

we compute the complete one-loop diagram, while keeping the external line corresponding to k and k' off-shell, and putting the momenta p and p' on-shell. The resulting expression has the following form:

$$\langle kk'|\hat{S}|pp'\rangle\big|_{g^2} = \left[2ig\left(pp'+kk'\right)(2\pi)^2\delta^{(2)}(p+p'-k-k')\right]ig\frac{2pp'}{p-p'}(pp'+kk') \tag{2.21}$$

This already includes the sum of all one-loop diagrams with all possible placement of derivatives on internal and external lines, with the correct combinatorial factors. Noting that the term inside the square brackets in (2.21) coincides with the expression for the interaction vertex (2.20), one can regard the one-loop scattering amplitude as the interaction vertex in momentum representation, with the momenta p and p' on-shell, multiplied by some function of this pair of momenta and the expansion parameter. So that the n-loop scattering amplitude corresponds to the product of n of these modified vertices.

$$\langle kk'|\hat{S}|pp'\rangle\big|_{g^{n+1}} = \left(ig\frac{pp'}{p-p'}\right)^n 2ig(2\pi)^2\delta^{(2)}(p+p'-k-k')(pp'+kk') \tag{2.22}$$

If we also put k and k' on-shell, we find that:

$$\langle kk'|\hat{S}|pp'\rangle\big|_{g^{n+1}} = 2\left(ig\frac{pp'}{p-p'}\right)^{n+1} \delta_+^{(2)}(p,p';k,k'), \tag{2.23}$$

where we have used the relation

$$(2\pi)^2\delta^{(2)}(p+p'-k-k') = \frac{1}{2(p-p')}\delta_+^{(2)}(p,p';k,k') \tag{2.24}$$

Thus, two-particle scattering S-matrix has the form [27].

$$S(p,p') = 1 + 2\sum_{n=1}^{\infty}(ig)^n\left(\frac{pp'}{p-p'}\right)^n = \frac{\frac{1}{p}-\frac{1}{p'}-ig}{\frac{1}{p}-\frac{1}{p'}+ig} \xrightarrow{g\to 1} \frac{\frac{1}{p}-\frac{1}{p'}-i}{\frac{1}{p}-\frac{1}{p'}+i} \tag{2.25}$$

3. Three-particle scattering S-matrix

The direct verification of the S-matrix factorization using the perturbative calculations is quite difficult to carry out, due to the complicated diagrammatic analysis. The only model for which the factorization property was manifestly verified is the non-linear Schrdinger model. We refer to the details of the diagrammatic calculations for the case of the NLS model to [1, 2]. Before presenting the concrete calculations for the LL model let us note that there is a significant difference between the NLS and LL models, and the latter case is considerably more complex. Indeed, the Hamiltonian for the NLS model has the form:

$$H = \int dx\ |\partial_x\phi(x)|^2 + c|\phi(x)|^4 \tag{3.1}$$

Thus, while calculating the three-particle scattering amplitude, one has to deal with only one type of vertex. The same is true when considering the general N-particle scattering process. In the LL model the situation is different and to consider the three-particle scattering amplitude one has to truncate the full action (2.10) up to the sixth order in fields. The higher order vertices will not contribute as the consequence of the retarded propagator as well as the charge conservation [27]. The resulting Lagrangian has the form

$$\begin{aligned}\mathcal{L} = {}& \frac{i}{2}(\varphi^*\partial_t\varphi - \partial_t\varphi^*\varphi) - |\partial_x\varphi|^2 - \frac{g_1}{2}[(\varphi^*\partial_x\varphi)^2 + (\partial_x\varphi^*\varphi)^2] \\ & -\frac{g_2}{4}|\varphi|^2[(\varphi^*\partial_x\varphi)^2 + (\partial_x\varphi^*\varphi)^2] - \frac{g_3}{2}|\varphi|^4|\partial_x\varphi|^2\end{aligned} \tag{3.2}$$

where we have introduced, analogous to the two-particle scattering perturbative calculations, arbitrary coupling constants g_1, g_2, g_3 for each type of vertices to keep track of perturbative calculations. As in the two-particle scattering case, we should take the limit

$g_i \to 1$, $i = 1,2,3$ in the end. As we will show below, this is a crucial point, making the S-matrix factorization possible. Clearly, dealing with more vertices in this case is not an easy task already for the three-particle case even at the one-loop level. As the number of particles considered in the scattering increases, one has to expand the action (2.10) to higher order, which, as a result, makes the complexity of the perturbative calculations and the combinatorial analysis practically impossible. Nevertheless, we make the first step in this direction and show in the next section the three-particle S-matrix factorization in the first order of perturbation.

3.1 Diagrammatic calculations

Assuming analyticity of the three-particle scattering amplitude in the coupling constants g_i it can be written as:[2]

$$\langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle = \langle \mathbf{k}|\mathbf{p}\rangle + \langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g + \langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2} + \cdots \tag{3.3}$$

where $\mathbf{k} = \{k_1, k_2, k_3\}$, and g is either of g_i, $i = 1,2,3$. The non-scattering term has the form:

$$\langle \mathbf{p}|\mathbf{k}\rangle = 3!(2\pi)^3 \mathcal{S}_p[\delta(p_1 - k_1)\delta(p_2 - k_2)\delta(p_3 - k_3)], \tag{3.4}$$

while the tree-level part yields:

$$\langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g = 9i(2\pi)^2 \mathcal{S}_{k,p}\left\{2g_1(k_1k_2 + p_1p_2)2\pi\delta(k_3 - p_3) + g_2(k_1k_2 + p_1p_2) - 2g_3k_1p_1\right\}\delta E\delta P \tag{3.5}$$

We have used here the symmetrization operator defined by [1]:

$$\mathcal{S}_p\left[f(\mathbf{p})\right] = \frac{1}{3!}\sum_A f(A\mathbf{p}) \tag{3.6}$$

where the sum is taken over all possible permutations of $(1,2,3)$, and the vector $A\mathbf{p} = (p_{A_1}, p_{A_2}, p_{A_3})$ is the corresponding set of momenta.[3] We have also introduced δE and δP to denote the delta functions for the total energy and momentum conservation, i.e.,

$$\delta E = \delta\left(\sum_{i=1}^{3}(k_i^0 - p_i^0)\right) \tag{3.7}$$

$$\delta P = \delta\left(\sum_{i=1}^{3}(k_i - p_i)\right) \tag{3.8}$$

The tree level Feynman graphs (connected and disconnected) in the first order are pictured in figure 2, where the spatial derivatives are represented by the marks over the particle lines. We stress that each of these graphs denotes a sum over diagrams of the same topology but with different permutations of the external momenta.

The expression for the tree level (3.5) defines the interaction vertex in the momentum representation, represented by figure 3.

[2]In the case of the NLS model, the analyticity of the S-matrix can be seen from the N-particle coordinate Bethe ansatz calculations of Yang [64, 65]. In the case of the LL model it is a natural assumption.

[3]Without any loss of generality, we choose the momenta $\mathbf{p}$ to be arranged in the order $p_1 > p_2 > p_3$.

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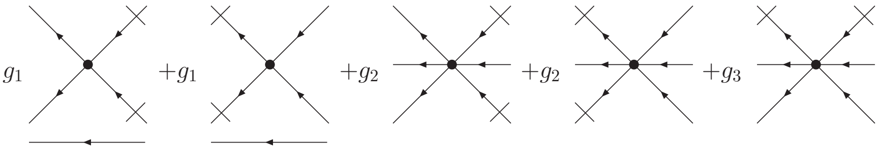

Figure 2: The tree-level Feynman diagrams for the three-particle scattering in the Landau-Lifshitz model.

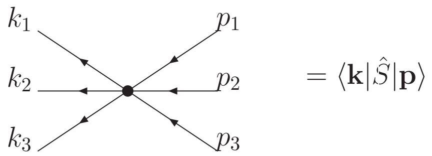

Figure 3: Diagrammatic representation of the interaction vertex in momentum space.

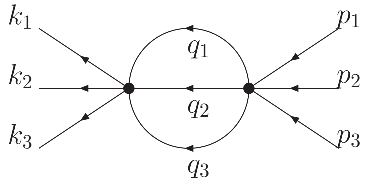

Figure 4: Diagrammatic representation of the second order scattering amplitude.

Using the interaction vertex in the momentum representation we are able to compute the second order scattering amplitude as depicted in figure 4.

The analytical expression corresponding to the figure 4 is given by:

$$
\begin{aligned}
\langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2} =& \frac{1}{6}\int\prod_{j=1}^{3}\left[\frac{d^2 q_j}{4\pi^2}\frac{i}{q_j^0 - q_j^2 + i\epsilon}\right] 9i(2\pi)^2 \mathcal{S}_{k,q}\{2g_1(k_1k_2+q_1q_2)2\pi\delta(k_3-q_3)+ \\
&+g_2(k_1k_2+q_1q_2) - 2g_3k_1q_1\}\,\delta\left(\sum_{i=1}^{3}(k_i^0-q_i^0)\right)\delta\left(\sum_{i=1}^{3}(k_i-q_i)\right)\; 9i(2\pi)^2\mathcal{S}_{q,p}\,\{2g_1(q_1q_2+p_1p_2) \\
&\times 2\pi\delta(q_3-p_3) + g_2(k_1k_2+p_1p_2) - 2g_3k_1p_1\} \\
&\times\delta\left(\sum_{i=1}^{3}(q_i^0-p_i^0)\right)\delta\left(\sum_{i=1}^{3}(q_i-p_i)\right)
\end{aligned}
\tag{3.9}
$$

The delta functions in (3.9) may be used to integrate over q_3^0 and q_3, and the integrals over q_1^0 and q_2^0 can be performed by choosing the contour closing in the lower half-plane. We also symmetrize over q, in order to obtain an analytical expression depending only on the external symmetrized momenta,

$$
\begin{aligned}
\langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2} =& -\frac{3i}{2}\delta E\delta P\mathcal{S}_{p,k}\left\{\int\frac{dq_1dq_2}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon}\Big[\, 16\pi^2 g_1^2 f_1(\mathbf{k},\mathbf{q})f_1(\mathbf{p},\mathbf{q})+\right. \\
&+g_2^2 f_2(\mathbf{k},\mathbf{q})f_2(\mathbf{p},\mathbf{q}) + 4g_3^2k_1p_1\sum k\sum p + 4\pi g_1g_2(f_1(\mathbf{k},\mathbf{q})f_2(\mathbf{p},\mathbf{q}) + f_2(\mathbf{k},\mathbf{q})f_1(\mathbf{p},\mathbf{q})) -
\end{aligned}
$$

$$
\begin{aligned}
&- 8\pi g_1 g_3 \left(p_1 \sum p f_1(\mathbf{k},\mathbf{q}) + k_1 \sum k f_1(\mathbf{p},\mathbf{q})\right) - \\
&- 2 g_2 g_3 \left(p_1 \sum p f_2(\mathbf{k},\mathbf{q}) + k_1 \sum k f_2(\mathbf{p},\mathbf{q})\right) \Big]\Big\}
\end{aligned} \tag{3.10}
$$

where we introduced the following functions:

$$
\begin{aligned}
f_1(\mathbf{x},\mathbf{q}) =&(x_1 x_2 + q_1 q_2)\delta(x_1 + x_2 - q_1 - q_2) + \left[x_1 x_2 + q_1\left(\sum x - q_1 - q_2\right)\right]\delta(x_3 - q_2) + \\
&+ \left[x_1 x_2 + q_2\left(\sum x - q_1 - q_2\right)\right]\delta(x_3 - q_1)
\end{aligned} \tag{3.11}
$$

$$
f_2(\mathbf{x},\mathbf{q}) = 3 x_1 x_2 + (q_1 + q_2)\sum x - q_1^2 - q_2^2 - q_1 q_2 \tag{3.12}
$$

To avoid cluttering we do not write the indices in the sum, and use the notation $\sum x \equiv \sum_{i=1}^{3} x_i$.

It is easier to compute each term of (3.10) separately.

(a) Term proportional to g_1^2:

Most of the integrals are trivially calculated due to the delta functions in $f_1(\mathbf{x},\mathbf{q})$, and the remaining delta functions corresponding to the total energy and momentum conservation allow us to write these terms as a sum of the following contributions:

(i) the finite part, which is a function of the external momenta, and

(ii) the term proportional to $\delta(p_3 - k_3)$.

It is the latter term that may lead to divergencies, which, however, could be regularized in the same manner as in [27]. Indeed, we notice that this term can be written in the following form:

$$
-i k_1 k_2 p_1 p_2 I_0(k_1,k_2) + i(p_1 p_2 + k_1 k_2) I_1(k_1,k_2) - i I_2(k_1,k_2),
$$

where the one-loop integrals I_0, I_1 and I_2 were regularized and explicitely calculated in [27]. (We refer the reader to [27] for definitions and all calculational details). The finite result after this regularization has the form (where we keep here the integral representation to be dealt only in the end of the calculations):

$$
\begin{aligned}
&96\pi^2 g_1^2 \frac{[k_1 k_2 + p_3(k_1 + k_2 - p_3)][p_1 p_2 + k_3(p_1 + p_2 - k_3)]}{\sum p^0 - p_3^2 - k_3^2 - (p_1 + p_2 - k_3)^2 + i\epsilon} - \\
&- 24\pi^2 g_1^2 \int dq \frac{[s - 2p_1 p_2]\left[s + \frac{1}{4}(p_1 - p_2)^2 - \frac{1}{4}(k_1 - k_2)^2 - 2k_1 k_2\right]}{q^2 - \frac{1}{4}(p_1 - p_2)^2 - s - i\epsilon}\delta(k_3 - p_3)
\end{aligned} \tag{3.13}
$$

We have denoted $s \equiv \frac{1}{2}\left(\sum p^0 - \sum p^2\right)$ and shifted the integration variable $q \to q + 1/2(k_1 + k_2)$. The corresponding diagrams are presented in figure 5. Once again, we emphasize that each diagram corresponds to the sum over all diagrams with the same topology, but with distinct permutations over the external momenta and different placements of the derivatives. The first diagram in figure 5 corresponds to the first term in (3.13), and the second term is the finite part of the second diagram.

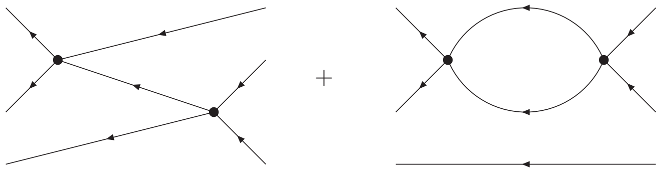

Figure 5: Feynman diagram corresponding to the equation (3.13).

The diagrams, corresponding to 5, but with explicit placement of the derivatives are depicted on figures 6 and 7 respectively.[4]

(b) Term proportional to g_2^2:

After some algebraic manipulations this term can be written in the form:

$$
\begin{aligned}
& g_2^2 \int \frac{f_2(\mathbf{k},\mathbf{q}) f_2(\mathbf{p},\mathbf{q})\; dq_1 dq_2}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon} \\
& = -\frac{g_2^2}{2} \int dq_1 dq_2 \frac{[3p_1p_2 - s]\,[3k_1k_2 - s]}{q_1^2 + q_2^2 + q_1q_2 - (q_1+q_2)\sum p - s - i\epsilon}
\end{aligned}
\tag{3.14}
$$

The corresponding Feynman diagram is represented by figure 8.

(c) Term proportional to g_3^2:

Simple transformations lead to the following expression:

$$
\begin{aligned}
& 4g_3^2 \int \frac{k_1 p_1 \sum k \sum p\; dq_1 dq_2}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon} \\
& = -2g_3^2 \int \frac{k_1 p_1 \sum k \sum p\; dq_1 dq_2}{q_1^2 + q_2^2 + q_1q_2 - (q_1+q_2)\sum p - s - i\epsilon}
\end{aligned}
\tag{3.15}
$$

The corresponding Feynman diagram is represented by figure 9.

(d) Term proportional to g_1g_2:

One of the integrals in this term is trivially computed using the delta functions of $f_1(\mathbf{x},\mathbf{q})$, while the remaining integral after some algebraic manipulations and the shift of the integration variable $q_1 \to q_1 + 1/2(k_1+k_2)$ and $q_2 \to q_2 + 1/2(p_1+p_2)$, yields:

$$
\begin{aligned}
& 4\pi g_1 g_2 \int dq_1 dq_2\, \frac{f_1(\mathbf{k},\mathbf{q}) f_2(\mathbf{p},\mathbf{q}) + f_2(\mathbf{k},\mathbf{q}) f_1(\mathbf{p},\mathbf{q})}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon} \\
& = -6\pi g_1 g_2 \int dq\, \frac{[k_3(k_1+k_2) - k_1k_2 + s]\,[s - 3p_1p_2]}{q^2 - \frac{1}{4}(k_1+k_2)^2 - k_3(k_1+k_2) - s - i\epsilon} - \\
& - 6\pi g_1 g_2 \int dq\, \frac{[p_3(p_1+p_2) - p_1p_2 + s]\,[s - 3k_1k_2]}{q^2 - \frac{1}{4}(p_1+p_2)^2 - p_3(p_1+p_2) - s - i\epsilon}
\end{aligned}
\tag{3.16}
$$

The corresponding Feynman diagram is represented by figure 10.

[4]We draw the remaining Feynman diagrams in the appendix.

(e) Term proportional to $g_1 g_3$:

Once again the delta functions in $f_1(\mathbf{x},\mathbf{q})$ allow us to trivially compute one of the integrals and the resulting expression has the form:

$$\begin{aligned}
&-8\pi g_1 g_3 \int dq_1 dq_2\, \frac{p_1 \sum p f_1(\mathbf{k},\mathbf{q}) + k_1 \sum k f_1(\mathbf{p},\mathbf{q})}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon} \\
&= -12\pi g_1 g_3 \int dq \left\{p_1 \sum p + k_1 \sum k\right\} - \\
&\quad - 12\pi g_1 g_3 \int dq \frac{k_1 \sum k\, [p_3(p_1+p_2) - p_1 p_2 + s]}{q^2 - \frac{1}{4}(p_1+p_2)^2 - p_3(p_1+p_2) - s - i\epsilon} \\
&\quad - 12\pi g_1 g_3 \int dq \frac{p_1 \sum p\, [k_3(k_1+k_2) - k_1 k_2 + s]}{q^2 - \frac{1}{4}(k_1+k_2)^2 - k_3(k_1+k_2) - s - i\epsilon}
\end{aligned} \tag{3.17}$$

The corresponding Feynman diagram is represented by figure 11.

(f) Term proportional to $g_2 g_3$:

There are no delta functions present in this term and after some transformations we write it in the form:

$$\begin{aligned}
&-2 g_2 g_3 \int \frac{p_1 \sum p f_2(\mathbf{k},\mathbf{q}) + k_1 \sum k f_2(\mathbf{p},\mathbf{q})}{\sum p^0 - q_1^2 - q_2^2 - (\sum p - q_1 - q_2)^2 + i\epsilon} dq_1 dq_2 \\
&= -g_2 g_3 \int dq_1 dq_2 \frac{(p_1 \sum p + k_1 \sum k)\, s - 3k_1 k_2 p_1 \sum p - 3k_1 p_1 p_2 \sum k}{q_1^2 + q_2^2 + q_1 q_2 - (q_1+q_2)\sum p - s - i\epsilon}
\end{aligned} \tag{3.18}$$

The corresponding Feynman diagram represented by figure 12.

Let us note, that in equations (3.13), (3.14), (3.16), (3.17) and (3.18) we have ignored the divergences and kept only the finite parts. Although the above formulas have been obtained for the general off-shell case, putting the external momenta on shell, i.e., $p_i^0 = p_i^2$ and $k_i^0 = k_i^2$, for $i = 1,2,3$ leads to considerable simplifications. This is due to the following identities:

$$\begin{aligned}
&\sum p^0 - \sum p^2 = \sum k^0 - \sum k^2 = 0 \\
&\sum p - \left(\sum p\right)^2 = -2(p_1 p_2 + p_1 p_3 + p_2 p_3), \sum k - \left(\sum k\right)^2 = -2(k_1 k_2 + k_1 k_3 + k_2 k_3) \\
&\sum p - \left(\sum p\right)^2 = \sum k - \left(\sum k\right)^2 \Rightarrow p_1 p_2 + p_1 p_3 + p_2 p_3 = k_1 k_2 + k_1 k_3 + k_2 k_3
\end{aligned} \tag{3.19}$$

The remaining integrals are not difficult to compute. In fact, the integral that one has to compute in the second term of (3.13) is simply,

$$\int \frac{dq}{q^2 - \frac{1}{4}(p_1 - p_2)^2 - i\epsilon} = \frac{2\pi i}{|p_1 - p_2|} \tag{3.20}$$

The remaining integrals in (3.16) and (3.17) are:

$$\int \frac{dq}{q^2 - \frac{1}{4}(k_1+k_2)^2 - k_3(k_1+k_2) + k_1 k_2 + k_1 k_3 + k_2 k_3 - i\epsilon} = \frac{2\pi i}{|k_1 - k_2|} \tag{3.21}$$

$$\int \frac{dq}{q^2 - \frac{1}{4}(p_1+p_2)^2 - p_3(p_1+p_2) + p_1 p_2 + p_1 p_3 + p_2 p_3 - i\epsilon} = \frac{2\pi i}{|p_1 - p_2|} \tag{3.22}$$

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Finally the remaining double integral in (3.14), (3.15) e (3.18) is:

$$\begin{aligned}\int \frac{dq_1 dq_2}{q_1^2+q_2^2+q_1q_2-p(q_1+q_2)+(p_1p_2+p_1p_3+p_2p_3)-i\epsilon} \\ = -\frac{4\pi}{\sqrt{3}}\left[\frac{i\pi}{2}-\frac{1}{2}\log Q^2+\frac{1}{2}\lim_{R\to\infty}\log(R^2-Q^2)\right]\end{aligned} \tag{3.23}$$

where we denoted:

$$Q^2=\frac{4}{6}\left[(p_1-p_2)^2+(p_1-p_3)^3+(p_2-p_3)^2\right] \tag{3.24}$$

Keeping only the finite part in the integral (3.23), and collecting the above expressions, we find that the second order scattering amplitude has the form:

$$\begin{aligned}\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2} = -\frac{3}{2}\mathcal{S}_{p,k}\Bigg\{96\pi^2g_1^2\Bigg[&\frac{[k_1k_2+p_3(k_1+k_2-p_3)][p_1p_2+k_3(p_1+p_2-k_3)]}{p_1^2+p_2^2-k_3^2-(p_1+p_2-k_3)^2+i\epsilon} \\ &-\frac{k_1k_2p_1p_2}{p_1-p_2}2\pi i\delta(p_3-k_3)\Bigg]+ \\ +k_1p_1\Bigg[&48\pi^2ig_1g_3\left(\frac{k_1p_2}{p_1-p_2}+\frac{k_2p_1}{|k_1-k_2|}\right) \\ &+\frac{4\pi}{\sqrt{3}}(i\pi-\log Q^2)\left((k_1p_2+k_2p_1)g_3(2g_3-3g_2)+g_3^2k_1p_1\right)\Bigg]+ \\ +k_1k_2p_1p_2\Bigg[&48\pi^2i\left(\frac{1}{p_1-p_2}+\frac{1}{|k_1-k_2|}\right)g_1(2g_3-3g_2) \\ &+\frac{4\pi}{\sqrt{3}}(i\pi-\log Q^2)(2g_3-3g_2)^2\Bigg]\Bigg\}\,\delta E\delta P\end{aligned} \tag{3.25}$$

3.2 The S-matrix factorization

Had we assumed the quantum integrability of the LL model, we could have obtained the S-matrix for three particle scattering by using the relation:

$$\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle = S(\mathbf{p})\delta_+^{(3)}(\mathbf{p},\mathbf{k}) \tag{3.26}$$

where $\delta_+^{(3)}(\mathbf{p},\mathbf{k}) = \langle\mathbf{p}|\mathbf{k}\rangle = 3!(2\pi)^3\mathcal{S}_p[\delta(p_1-k_1)\delta(p_2-k_2)\delta(p_3-k_3)]$. This relation, analogous to (2.15), underlines the fact that in the scattering process the particle annihilation and creation are not possible, and the set of momenta before and after the scattering is the same. This is not obvious from the calculations presented in the previous section. Indeed, although the non-scattering term clearly has the form (3.26), neither the tree level (3.5), nor the second order (3.25) terms are manifestly of this form. Nevertheless, we will show that this is indeed the case, and one can reduce the expressions for the tree level (3.5) and the second order (3.25) contributions to the forms that are proportional to $\delta_+^{(3)}(\mathbf{p},\mathbf{k})$.

Let us first focus on the tree level term (3.5). Expanding explicitly the symmetrization operator $\mathcal{S}_{k,p}$, we find:

$$\begin{aligned}\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_g =& 2ig_1\left[\frac{p_1p_2}{p_1-p_2}+\frac{p_1p_3}{p_1-p_3}+\frac{p_2p_3}{p_2-p_3}\right]\delta_+^{(3)}(\mathbf{p},\mathbf{k})+ \\ &+2i(2\pi)^2\left\{(p_1p_2+p_1p_3+p_2p_3)(3g_2-2g_3)-g_3(p_1^2+p_2^2+p_3^2)\right\}\delta E\delta P\end{aligned} \tag{3.27}$$

We immediately notice that the two terms in above expression are quite different. Namely, the first term of (3.27) is already in the form needed for the S-matrix to satisfy (3.26), while the second term, on the other hand, does not satisfy this condition. Therefore, had we restricted ourselves only to this order, the condition (3.26), necessary for integrability to hold, would have failed. Similarly, it is not difficult to see that the second order scattering amplitude also contains terms that are not proportional to $\delta_+^{(3)}(\mathbf{p}, \mathbf{k})$, such as the terms in the second and third square brackets in (3.25).

Let us note, however, that if the quantum integrability, and as a consequence, the S-matrix factorization were true, the three-particle S-matrix would have the exact form:

$$\begin{aligned} S^{(3)}(\mathbf{p}, \mathbf{k}) &= S^{(2)}(p_1, p_2) S^{(2)}(p_1, p_3) S^{(2)}(p_2, p_3) \\ &= \left(\frac{\frac{1}{p_1} - \frac{1}{p_2} - ig}{\frac{1}{p_1} - \frac{1}{p_2} + ig} \right) \left(\frac{\frac{1}{p_1} - \frac{1}{p_3} - ig}{\frac{1}{p_1} - \frac{1}{p_3} + ig} \right) \left(\frac{\frac{1}{p_2} - \frac{1}{p_3} - ig}{\frac{1}{p_2} - \frac{1}{p_3} + ig} \right) \\ &= 1 + 2 \sum_{n=1}^{2} \left[ig \left(\frac{p_1 p_2}{p_1 - p_2} + \frac{p_1 p_3}{p_1 - p_3} + \frac{p_2 p_3}{p_2 - p_3} \right) \right]^n + O(g^3) \end{aligned} \tag{3.28}$$

where we have expanded in the last line the exact expression up to the second order in g. Comparing the equations (3.27) and (3.28), we see that the unity in (3.28) corresponds to the non-scattering part $\langle \mathbf{p} | \mathbf{k} \rangle$, while the first term in (3.27) corresponds to the first order term of (3.28). We will now demonstrate, that the second term in (3.27) is such that it can be canceled out exactly with a term from the second order contribution (3.25).

A very similar expression for first term of (3.25) appears in the NLS model (see formula (13) of [1]). In the latter case, the use of the formula

$$\frac{1}{x \pm i0} = \mp i\pi\delta(x) + \text{P.V.} \left(\frac{1}{x} \right) \tag{3.29}$$

proved to be crucial to show the factorization (see also [53]). We, thus, expect that the usage of this formula should play the same role in our case. However, while the principal value part has no contribution for the NLS model, in the LL model it plays a non-trivial role and leads to the cancellations which make the factorization possible. We draw all the Feynman graphs for this term in figure 6, where only the sum over topologically similar diagrams with different permutations of the external momenta is implicit. Thus, there are four diagrams, which correspond to all possible placements of the derivatives.

We remind that we have chosen the momenta $\mathbf{p}$ to be arranged in the order $p_1 > p_2 > p_3$ for the three-particle scattering to be possible. The principal value of the first term in (3.25) can be computed if we take $k_i \neq p_j$, $i, j \in \{1, 2, 3\}$ and use the identity

$$\frac{i}{k_i^2 + k_j^2 - p_l^2 - (k_i + k_j - p_l)^2 + i\epsilon} = \frac{1}{2} \frac{i}{k_i - k_j} \left(\frac{1}{p_l - k_j - i\epsilon} + \frac{1}{k_i - p_l - i\epsilon} \right),$$

as well as take into account the delta functions corresponding to the energy and momentum conservation. Only the first and last diagrams on the R.H.S of the figure 6 will contribute

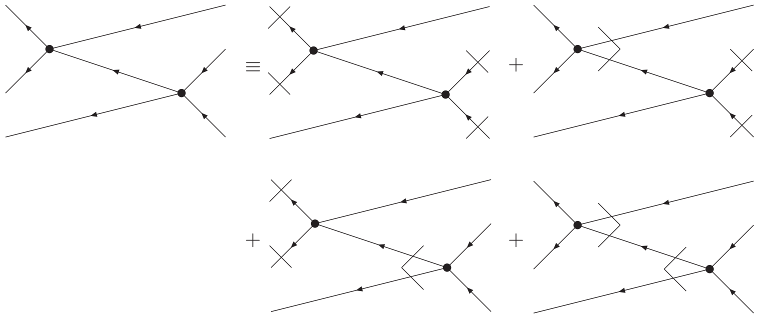

Figure 6: The Feynman diagrams for the first term in the equation (3.25) with all derivatives placed.

to the principal value, while the other two vanish identically due to the momentum conservation. After some tedious transformations one finds:

$$\mathrm{P.V.}\left\{\mathcal{S}_{p,k}\left[\frac{[k_1k_2+p_3(k_1+k_2-p_3)][p_1p_2+k_3(p_1+p_2-k_3)]}{p_1^2+p_2^2-k_3^2-(p_1+p_2-k_3)^2+i\epsilon}\right]\right\}=\frac{1}{18}(-p_1^2-p_2^2-p_3^2+p_1p_2+p_1p_3+p_2p_3) \tag{3.30}$$

The delta function part of (3.29) yields:

$$\delta\left(p_1^2+p_2^2-k_3^2-(p_1+p_2-k_3)\right)=\frac{1}{2(p_1-p_2)}\left[\delta(p_1-k_3)+\delta(p_2-k_3)\right] \tag{3.31}$$

Substituting this result back into (3.25) we find:

$$\begin{aligned}\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2}=&8i\pi^2g_1^2(p_1^2+p_2^2+p_3^2-p_1p_2-p_1p_3-p_2p_3)\delta E\delta P-\\ &-\frac{3i}{2}\delta E\delta P\;\mathcal{S}_{p,k}\Bigg\{-96\pi^2g_1^2\frac{k_1k_2p_1p_2}{p_1-p_2}2\pi i\left[\delta(k_3-p_1)+\delta(k_3-p_2)+\delta(k_3-p_3)\right]\\ &\qquad+k_1p_1\Bigg[48\pi^2ig_1g_3\left(\frac{k_1p_2}{p_1-p_2}+\frac{k_2p_1}{|k_1-k_2|}\right)\\ &\qquad\qquad+\frac{4\pi}{\sqrt{3}}(i\pi-\log Q^2)\big((k_1p_2+k_2p_1)g_3(2g_3-3g_2)+g_3^2k_1p_1\big)\Bigg]+\\ &\qquad+k_1k_2p_1p_2\Bigg[48\pi^2i\left(\frac{1}{p_1-p_2}+\frac{1}{|k_1-k_2|}\right)g_1(2g_3-3g_2)\\ &\qquad\qquad+\frac{4\pi}{\sqrt{3}}(i\pi-\log Q^2)(2g_3-3g_2)^2\Bigg]\Bigg\}\end{aligned} \tag{3.32}$$

To proceed, we explicitly expand the symmetrization operator $\mathcal{S}_{k,p}$ and obtain the following form for the second order scattering amplitude:

$$\begin{aligned}\langle\mathbf{k}|\hat{S}|\mathbf{p}\rangle\Big|_{g^2}=&8i\pi^2g_1^2(p_1^2+p_2^2+p_3^2-p_1p_2-p_1p_3-p_2p_3)\delta E\delta P+\\ &+2i^2g_1^2\left[\frac{p_1p_2}{p_1-p_2}+\frac{p_1p_3}{p_1-p_3}+\frac{p_2p_3}{p_2-p_3}\right]^2\delta_+^{(3)}(\mathbf{p},\mathbf{k})-\frac{3i}{2}\mathcal{R}(g^2)\delta E\delta P\end{aligned} \tag{3.33}$$

where we have separated the one loop contribution:

$$\mathcal{R}(g^2) = -\frac{16}{3}\pi^2 i g_1 \left\{ \left[\frac{p_1p_2}{p_1-p_2} + \frac{p_1p_3}{p_1-p_3} + \frac{p_2p_3}{p_2-p_3} \right] \left[(3g_2-2g_3)(k_1k_2+k_1k_3+k_2k_3) - g_3(k_1^2+k_2^2+k_3^2) \right] + \right.$$
$$\left. + \left[\frac{k_1k_2}{|k_1-k_2|} + \frac{k_1k_3}{|k_1-k_3|} + \frac{k_2k_3}{|k_2-k_3|} \right] \left[(3g_2-2g_3)(p_1p_2+p_1p_3+p_2p_3) - g_3(p_1^2+p_2^2+p_3^2) \right] \right\} +$$
$$+ \frac{4\pi}{\sqrt{3}}(i\pi - \log Q^2) \left[(3g_2-2g_3)(p_1p_2+p_1p_3+p_2p_3) - g_3(p_1^2+p_2^2+p_3^2) \right] \times$$
$$\times \left[(3g_2-2g_3)(k_1k_2+k_1k_3+k_2k_3) - g_3(k_1^2+k_2^2+k_3^2) \right] \tag{3.34}$$

We note that in the limit $g_1 = g_2 = g_3 = 1$ the first term of (3.33) has the opposite sign of the second term in (3.27). Thus, the unwanted term indeed cancels out and the remaining terms combine into the form:

$$\langle \mathbf{k}|\hat{S}|\mathbf{p}\rangle = \left[1 + 2\sum_{n=1}^{2} \left[i\left(\frac{p_1p_2}{p_1-p_2} + \frac{p_1p_3}{p_1-p_3} + \frac{p_2p_3}{p_2-p_3} \right) \right]^n \right] \delta_+^{(3)}(\mathbf{p},\mathbf{k}) - \frac{3i}{2}\mathcal{R}(g^2)\delta E \delta P + O(g^3) \tag{3.35}$$

The first term in (3.35) is the three-particle scattering S-matrix in the second order, consistent with the factorization relation (3.28). Thus, we conclude that to have the S-matrix factorization already in the lowest order, the terms from the higher order contribution should be taken into account to cancel out the unwanted terms. Diagrammatically, by making use of the relation (3.29) we have found that the principal value of the Feynman diagrams depicted in figure 6 cancels out the terms at first order that prevented the factorizability of the S-matrix at this order, while the contribution of the delta-function term of (3.29), added to the term in figure 7, gave the factorizable second order contribution to the S-matrix. Although we left out the $\mathcal{R}(g^2)$ contribution in (3.35), this calculation provides us with a remarkable scheme in which the contributions of different orders cancel each other to yield S-matrix factorizability.

It is not difficult to see that the diagrams that we had to sum up to obtain the three-particle S-matrix in the first order are of the order $\hbar^0$ since neither the tree level, nor the diagrams in figure 6 contain any loops. Clearly, the cancellation at higher orders may happen for the diagrams of the same order in $\hbar$. Even though we were able to prove the factorizability at the first order, the complexity of diagrammatic and combinatorial analysis make the computations of higher orders practically impossible to carry out, and another approach has to be employed. This subject is currently under investigation.

4. Conclusion and discussion

In this paper we have shown the three-particle S-matrix factorization at the first order for the LL model. The calculations we have presented show the non-trivial mechanism behind the factorization even in the lowest order of perturbation. In the process we have also shown the absence of particle annihilation and creation in the scattering processes, and that the set of momenta before and after the scattering, correspondingly, is unchanged. Our consideration was motivated by the difficulties to analyze the quantum integrability of the LL model in the framework of the quantum inverse scattering method. The highly singular nature of the LL model does not allow the use of the trace identities and, as a result, the set of local operators in involution is in general hard to derive. In fact, only the action of the quantum Hamiltonian on the two-particle sector is known. In addition, the standard two-particle state (1.1), underlining the clear meaning of the scattering process, is not valid for the LL model quantized in terms of the fields $\varphi(x)$ (2.9), namely, the particles created by the operator $\varphi^+(x)$ are not the Bethe particles. As the S-matrix factorization, which expresses quantum integrability, relies on the presence of an infinite set of commuting local operators, it is important to independently verify quantum integrability using field theoretic methods. Let us note, that unlike the LL model, these subtleties are absent in the classically equivalent NLS model, and the quantum inverse scattering method is easily shown to be consistent with the field theoretic calculations.

The next obvious problem is generalizing the above construction to all orders of perturbation, as well as showing the factorization for the N-particle scattering amplitude. This is not an easy task, as even in the lowest order the difficulties of diagrammatic and combinatorial nature are quite complex. The essential difference with the NLS model is the new vertices that appear in each order for the LL model. As we have shown, the factorization mechanism is such that the higher order terms cancel the unwanted terms via the use of the formula (3.29). Clearly, at higher orders the analysis will be too complicated to be carried out. Thus, an alternative framework is needed to perform the diagrammatic calculations. The latter have to be also carried out to understand the effects of renormalization. The reason is twofold. First, as we have shown, the coupling constants $g_i, i = 1, 2, 3$, which we had introduced by hand to keep track of the perturbative order, had to be set to unity for the three-particle S-matrix to be factorizable. Although we have done so without properly taking care of the renormalization, it is important to carry out all the necessary calculations to show the validity of this assumption. Secondly, it is clear that the singularities arising in the quantum inverse scattering method, when deriving the local commuting operators, should exhibit themselves in the renormalization of the coupling constants and fields. Thus, it would be interesting to establish a direct map between the two methods. We stress the importance of this program in the context of the AdS/CFT correspondence, since the difficulties present in the quantization of LL model will also be present in the higher sectors of the $AdS_5 \times S^5$ string.

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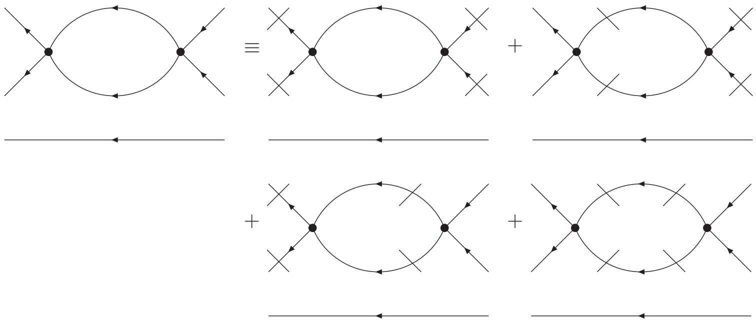


Figure 7: The Feynman diagrams with the correct placement of derivatives corresponding to the second Feynman graph pictured on figure 5.

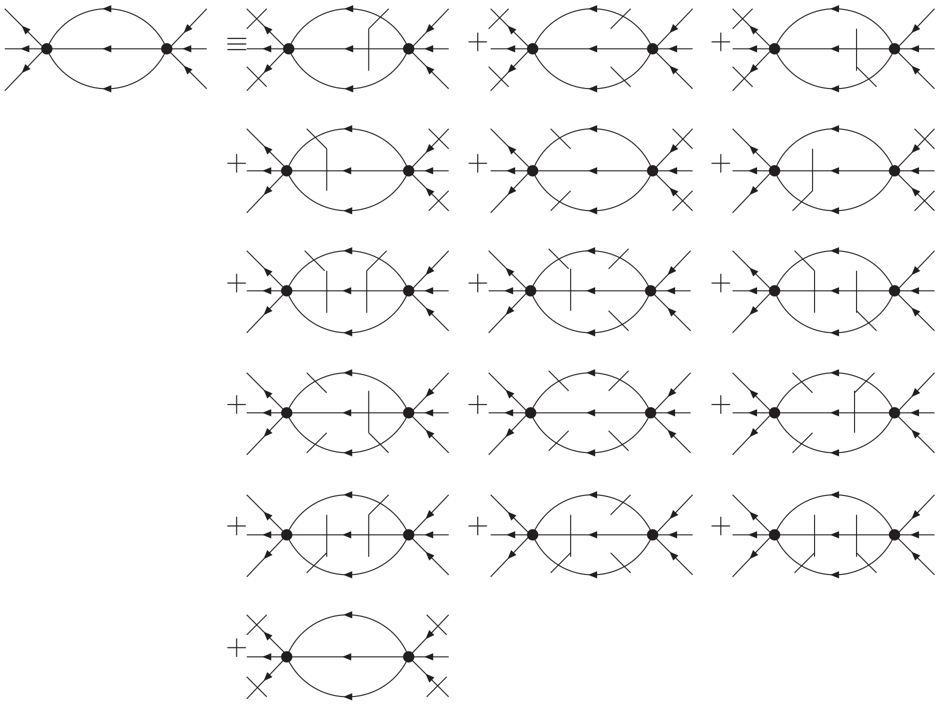


Figure 8: The Feynman diagrams proportional to g_2^2, corresponding to equation (3.14).

A. Feynman diagrams

In this appendix we draw the remaining Feynman graphs for the three-particle second order scattering amplitude with the explicit placement of derivatives. We note, however, that the sum over the diagrams of the same topology, but with different permutations of the external momenta, is assumed. The two diagrams depicted in figure 5, which correspond, respectively, to each of the finite terms of the equation (3.13), are represented, with all derivatives placed, by figures 6 and 7.

The following list of figures contains the complete Feynman diagrams corresponding to the equations (3.14)–(3.18).

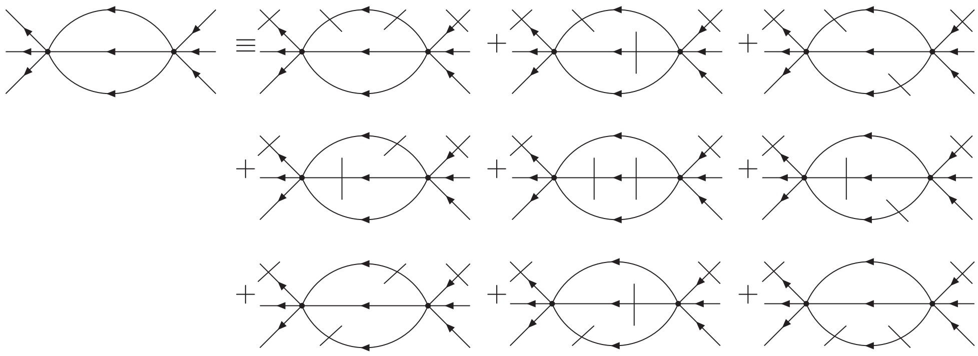
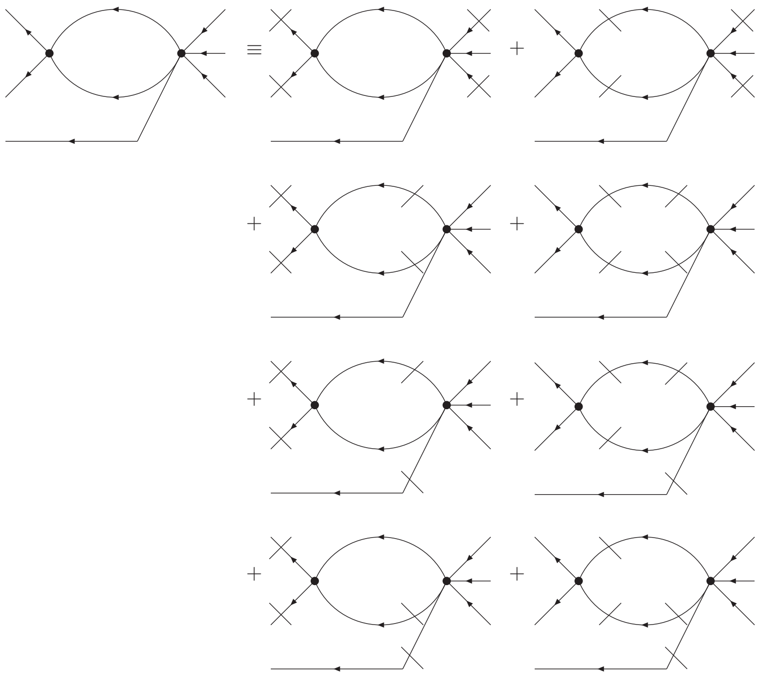


Figure 9: The Feynman diagrams proportional to g_3^2, corresponding to equation (3.15).



+ Time Reversed Diagrams

Figure 10: The Feynman diagrams proportional to $g_1 g_2$, corresponding to equation (3.16).

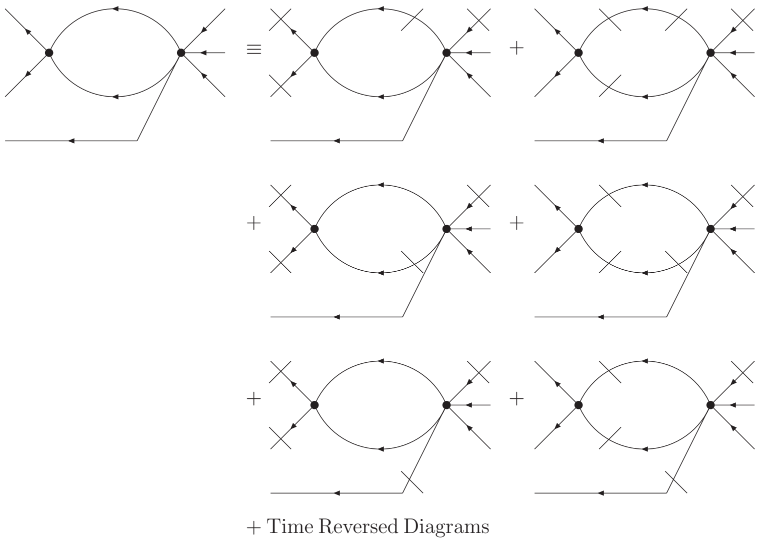


Figure 11: The Feynman diagrams proportional to $g_1 g_3$, corresponding to equation (3.17).

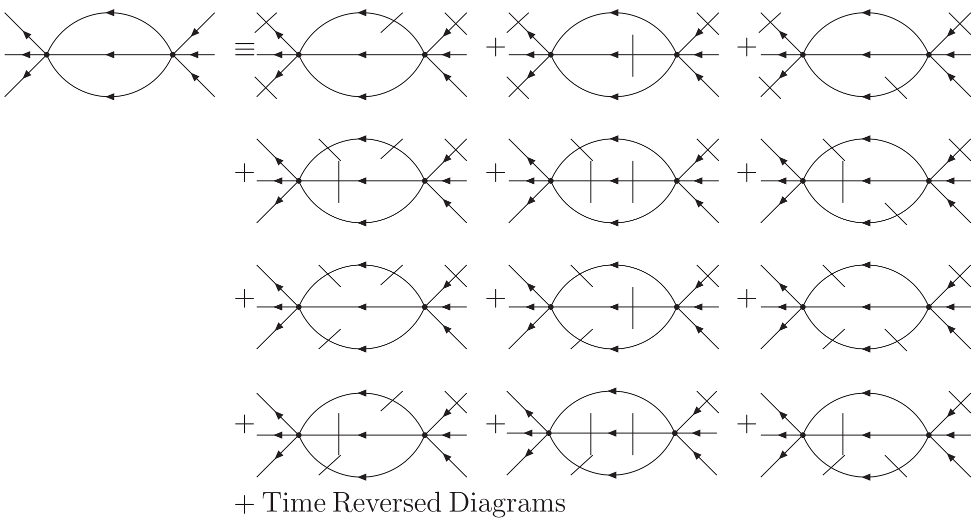


Figure 12: The Feynman diagrams proportional to $g_2 g_3$, corresponding to equation (3.18).

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